The Numerical Solution of Scalar Field for Nariai Case in 5D Ricci-flat SdS Black String Space with Polynomial Approximation

Chunxiao Wang,* Molin Liu,† and Hongya Liu‡
School of Physics and Optoelectronic Technology,
Dalian University of Technology, Dalian, 116024, P. R. China

As one exact candidate of the higher dimensional black hole, the 5D Ricci-flat Schwarzschild-de Sitter black string space presents something interesting. In this paper, we give a numerical solution to the real scalar field around the Nariai black hole by the polynomial approximation. Unlike the previous tangent approximation, this fitting function makes a perfect match in the leading intermediate region and gives a good description near both the event and the cosmological horizons. We can read from our results that the wave is close to a harmonic one with the tortoise coordinate. Furthermore, with the actual radial coordinate the waves pile up almost equally near the both horizons.

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I. INTRODUCTION

Our notion about the universe and the structure of the spacetime in which we live have been radically changed by the existence of additional spacelike dimensions in nature. It is well known that the hierarchy problem, which is why the characteristic scale of gravity ($M_P \sim 10^9 GeV$) is about 16 orders of magnitude larger than the Electro-Weak scale ($W_{EW} \sim 1TeV$), is solved successfully both in ADD model [1] and RS model [2] [3]. In the former [1], Arakani-Hamed et al. suggest that compact extra dimensions can be as large as submillimeter scale and our 4D world is a lower-dimensional brane where all the matter is concentrated. That is, except for gravitons and scalar particles without charges are allowed to propagate in the bulk, all the other standard model particles are compressed on the 3-brane. Soon after, Randall and Sundrum present a different proposal that a noncompact spacelike fifth dimension is permitted to exist. RS 2-brane model [2] with a small extra dimension and RS 1-brane model [3] with a large extra dimension are build up. As a result of the instability of the vacuum in a strong gravitational field, the black hole can radiate various particles [4]. People usually call it Hawking Radiation or Black Hole Radiation. Based upon this momentous theory, many works are explored about the entroy of black holes (see [5]-[9]), the quasi-normal models (for a review, see [10] [11] [12]), and so on.

The 5D Ricci-flat Schwarzschild-de sitter (SdS) black hole, which contains a induced cosmological constant, originates from the canonial 1-body solution in the background of the Space-Time-Matter (STM) theory ([13]-[19]). Recently, this solution is embedded into a brane world by using a conformial factor [20] and constitute a 5D Ricci-flat SdS black string space. This black string space is the RS 2-brane model. Surfaces of constant y, which is the fifth dimension coordinate, satisfy the Israel equations. A reflection symmetric domain wall is in this spacetime. The potential surrounding the black string has a quantized spectrum as well as a continuous one by considering the standing

*Electronic address: chunxiaowang1231@sina.com
†Electronic address: mlliudl@student.dlut.edu.cn
‡Electronic address: hyliu@dlut.edu.cn

wave condition between its two branes. Soon, with the tangent approximation, the real scalar field equation is solved numerically in paper [21]. Then the extreme case— Nariai black hole is studied in the later work [22]. The decay of scalar field is more and more fierce with the increasing induced cosmological constant. In this paper, we attempt to study Nariai case with a new approach — polynomial approximation.

Nariai solution was presented by Nariai in the 1950s [23] and discussed by many literatures as one of important cases in the singular spaces [24]. It is the exact solution to the Einstein equation with positive cosmological constant without a Maxwell field. Meanwhile, it has a topological structure of a (1+1)-dimensional de Sitter spacetime with a round 2-sphere of fixed radius, i.e. $dS_2 \times S^2$. Here we study the massless scalar field about this extreme case in the 5D Ricci-flat SdS black string space. Using a polynomial as the fitting function which is more accurate than previous tangent approximation [22], we solve the full boundary problem numerically.

This paper is organized as follows: in section II, we introduce the background of this model. The 5D black string space and the two horizons of the space are presented. In section III, assuming the scalar field is separable, we decompose the scalar field equation successfully. In section IV, the master propagating equation with Schrödinger-like form is obtained via the tortoise coordinate transformation . In section V, with the boundary conditions and the polynomial approximation, we obtain a full boundary value problem. Then the Schrödinger-like equation is solved numerically. We finish with a summary of our result in the last section. We use the metric signature with diagonal $(+,-,-,-,-)$ and put $\hbar$, c, and G equal to unity. Lower-case Greek letters μ, ν, $\ldots$ will be taken to run over $0,1,2,3$ as usual, while upper-case Latin letters $A, B, \ldots$ runs over all five coordinates $0,1,2,3,4$.

II. 5D RICCI-FLAT BLACK STRING SPACE

We start our analysis by presenting an exact 5D black hole solution. The line element of the 5D Ricci-flat SdS black hole takes the form [14]

$$dS^2 = \frac{\Lambda\xi^2}{3}\left[f\left(r\right)dt^2 - \frac{1}{f\left(r\right)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] - d\xi^2, \tag{1}$$

where

$$f\left(r\right) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2, \tag{2}$$

ξ is the open non-compact extra dimension coordinate, M is the mass of the black hole, and Λ is the induced cosmological constant. The original aim of solution (1), which is one of important results of STM theory, is used to solve the canonical 1-body problem which is the 1-body solution cannot be ruled out by solar system tests [25]. Considering the 5D uniqueness theorem, metric (1) is introduced to describe the field outside an object like the Sun.

This solution satisfies the 5D vacuum equation

$$R_{AB} = 0. \tag{3}$$

Therefore, we can deduce a notable character of this space from (1) and (3). That is, the space is empty (no matter) when viewing from the 5D. However, viewing from the 4D (corresponds to $\xi = constant$), the matter is induced from the empty 5D manifold. So there is no cosmological constant, but only an effective cosmological constant Λ induced from the fifth dimension.

Mathematically, the three solutions of

$$f(r) = 0 \tag{4}$$

can be expressed [15] as:

$$\begin{cases} r_c = \frac{2}{\sqrt{\Lambda}} \cos \eta, \\ r_e = \frac{2}{\sqrt{\Lambda}} \cos(120^\circ - \eta), \\ r_o = - (r_e + r_c) \end{cases} \tag{5}$$

where $\eta = \frac{1}{3} \arccos(-3M\sqrt{\Lambda})$ with $30^\circ \leq \eta \leq 60^\circ$, and Λ must satisfy $\Lambda M^2 \leq \frac{1}{9}$ to ensure there are acceptable solutions to (4). We have labeled the black hole horizon (event horizon) and the cosmological horizon with r_e and r_c respectively. $r_o = -(r_e + r_c)$ is a negative solution and has no actual significance.

The space (1) is bounded by the two horizons. The interval between them is decreased as increasing Λ. The Nariai black hole arises when two horizons close to each other. All through the paper we will consider this extreme case. Consequently, the induced cosmological constant is chosen $\Lambda = 0.11$ (where have taken $M = 1$) like the valuation in [26] [27].

Expression (2) can be rewritten as follows

$$f(r) = \frac{\Lambda}{3r}(r - r_e)(r_c - r)(r - r_o). \tag{6}$$

Redefining the fifth dimension through the following transform

$$\xi = \sqrt{\frac{3}{\Lambda}} e^{\sqrt{\frac{\Lambda}{3}} y}, \tag{7}$$

we can get the new form of line element (1) as

$$dS^2 = e^{2\sqrt{\frac{\Lambda}{3}} y} \left[f\left(r\right) dt^2 - \frac{1}{f\left(r\right)} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right) - dy^2 \right], \tag{8}$$

where y is the new fifth dimension coordination.

III. KLEIN-GORDON EQUATION OF MASSLESS SCALAR FIELD IN THE BULK

Now we turn our attention to a massless scalar filed ϕ in the 5D black string space. The field $\phi\left(t, r, \theta, \varphi, y\right)$ obeys the Klein-Gordon equation

$$\Box \phi = 0. \tag{9}$$

By using the field factorization

$$\phi = \frac{1}{\sqrt{4\pi\omega}} \frac{1}{r} R_\omega(r, t) L(y) Y_{lm}(\theta, \varphi) \tag{10}$$

and

$$R_\omega(r, t) \rightarrow \Psi_{\omega ln}(r) e^{-i\omega t}, \tag{11}$$

where ω is the energy of scalar particles. $R_\omega(r, t)$ is the radial function about (r, t), $Y_{lm}(\theta, \varphi)$ is the usual spherical harmonic function, and $L(y)$ is the fifth dimension wave function. We obtain a set of decoupled 5th dimensional and radial equations

$$\frac{d^2 L\left(y\right)}{dy^2} + \Lambda \sqrt{\frac{\Lambda}{3}} \frac{dL\left(y\right)}{dy} + \Omega L\left(y\right) = 0, \tag{12}$$

$$\left[-f\left(r\right) \frac{d}{dr} \left(f\left(r\right) \frac{d}{dr} \right) + V\left(r\right) \right] \Psi_{\omega ln}\left(r\right) = \omega^2 \Psi_{\omega ln}(r), \tag{13}$$

where Ω, l are the constants which are adopted to separate variables. The potential function have been defined as follows:

$$V(r) = f(r)\left[\frac{1}{r}\frac{df(r)}{dr} + \frac{l(l+1)}{r^2} + \Omega\right]. \tag{14}$$

According to RS 2-brane model, there are two branes with Z_2 symmetry in the 5D bulk. They have opposite and equal tensions, and one of them contains the standard model fields. The two branes can be be set at $y = 0$ and $y = y_1$ (where y_1 is the thickness of the bulk) respectively. Using the standing wave condition between the two branes, we can solve Eq. (12) and get the spectrum of Ω, which includes continuous part below $\frac{3}{4}\Lambda$ and discrete part above $\frac{3}{4}\Lambda$ (about detail analysis, refer to [20]). The quantum parameter is denoted by $\Omega_n (n = 1, 2, 3, \ldots)$:

$$\Omega_n = \frac{n^2\pi^2}{y_1^2} + \frac{3}{4}\Lambda \tag{15}$$

IV. TORTOISE COORDINATE AND MASTER PROPAGATING EQUATION

Here we use the tortoise coordinate, defined by the relation

$$x = \frac{1}{2M}\int \frac{dr}{f(r)}. \tag{16}$$

By using (2), expression (16) can be integrated and the tortoise coordinate can be written as

$$x = \frac{1}{2M}\left[\frac{1}{2K_e}\ln\left(1 - \frac{r}{r_e}\right) - \frac{1}{2K_c}\ln\left(1 - \frac{r}{r_c}\right) + \frac{1}{2K_o}\ln\left(1 - \frac{r}{r_o}\right)\right], \tag{17}$$

where the surface gravities K_i is defined as:

$$K_i = \frac{1}{2}\left|\frac{df}{dr}\right|_{r=r_i}. \tag{18}$$

So the master propagating Eq. (13) takes the Schrödinger-like form

$$\left[-\frac{d^2}{dx^2} + 4M^2V(r)\right]\Psi_{\omega nl}(r) = 4M^2\omega^2\Psi_{\omega nl}(r), \tag{19}$$

and the gravitational potential barrier $V(r)$ that a scalar particles sees while propagating from the cosmological to the black hole horizon or vice versa takes the form

$$V(r) = f(r)\left[\frac{1}{r}\frac{df(r)}{dr} + \frac{l(l+1)}{r^2} + \frac{n^2\pi^2}{y_1^2} + \frac{3}{4}\Lambda\right]. \tag{20}$$

As an illuminating example, Fig. 1 depicts the form of this barrier for $n = 1, 2, 3$.

V. A FULL BOUNDARY VALUE PROBLEM

A. Boundary Conditions and Polynomial Approximation

In order to solve the wave equation (19) numerically, we need the boundary conditions at the horizons. Since the metric function $f(r)$ vanishes at both horizons, r_e and r_c, so does the potential barrier that is proportional to the metric function. Therefore, in both of these regimes, the solution of Eq. (19) has the form of plane waves, that is $e^{\pm i2M\omega x}$. Being consider only real field there, we can choose the boundary conditions at both horizons as

$$\Psi_{\omega ln}(x) = \cos(2M\omega x). \tag{21}$$

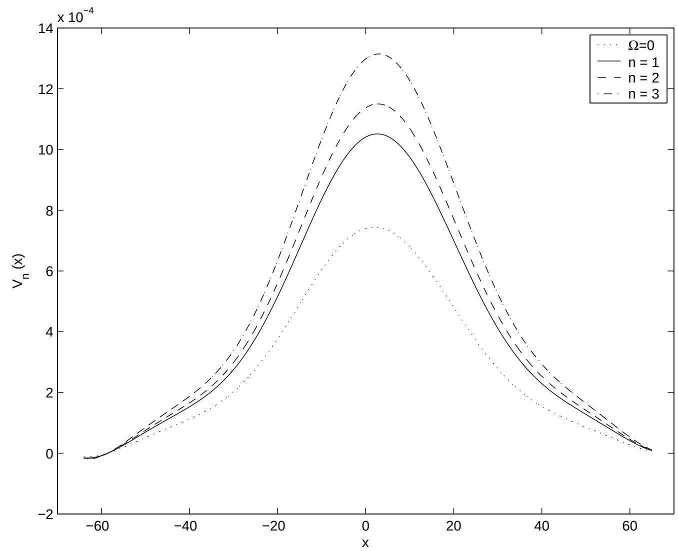

FIG. 1: The first 3 ($n = 1, 2, 3$) quantum potential barriers $V(r)$ for 5D string space. Here we selected $M = 1, \omega = 1, l = 1, \Lambda = 0.11$, and $y_1 = 10^{\frac{3}{2}}$ (a large extra dimension). The dotted line ($\Omega = 0$) corresponds to the 4D SdS case.

Before solve the Eq. (19), we need the potential is a function of of tortoise coordinate, i. e. $V = V(x)$. But it is difficult to invert Eq. (17) to get $r = r(x)$. So we can adopt the polynomial approximation

$$y = \tilde{r} = \sum_{i=0}^{10} a_i x^i \tag{22}$$

to approximate tortoise coordinate. The coefficients a_i are showed in Table I.

As an illustration of the accuracy of the approximation, we plot the curves of r and y (or $\tilde{r}$) versus x in Fig. 2, from which we notice that approximation (22) does not allow $|x|$ to become very large, so we shorten the interval of x to $[-70, 70]$, and the boundary condition (21) is rewritten as

$$\Psi_{\omega ln}(-70) = \Psi_{\omega ln}(70) = \cos(140M\omega). \tag{23}$$

It is to be noted here that the interval $[-70,\ 70]$ is selected according to the fitting situation. This setting can be illustrated clearly in Fig. 2. As far as we know, there is no way to give the accurate description near horizons in SdS black hole whatever in 4D or higher dimensional space. Comparing with the tangent approximation [27] [21] [22], the polynomial approximation shows good representing in despite of the shorter fitting interval $[-70,\ 70]$ than the origin one $[-100,\ 100]$ [27].

TABLE I: The coefficients in the approximating polynomial of degree 10

$a_0 = 2.9821$	$a_1 = 6.4016 \times 10^{-3}$	$a_2 = 3.4629 \times 10^{-5}$
$a_3 = -2.3878 \times 10^{-6}$	$a_4 = -1.7683 \times 10^{-8}$	$a_5 = 6.5626 \times 10^{-10}$
$a_6 = 4.5015 \times 10^{-12}$	$a_7 = -8.9336 \times 10^{-14}$	$a_8 = -5.5464 \times 10^{-16}$
$a_9 = 4.3675 \times 10^{-18}$	$a_{10} = 2.7548 \times 10^{-20}$	

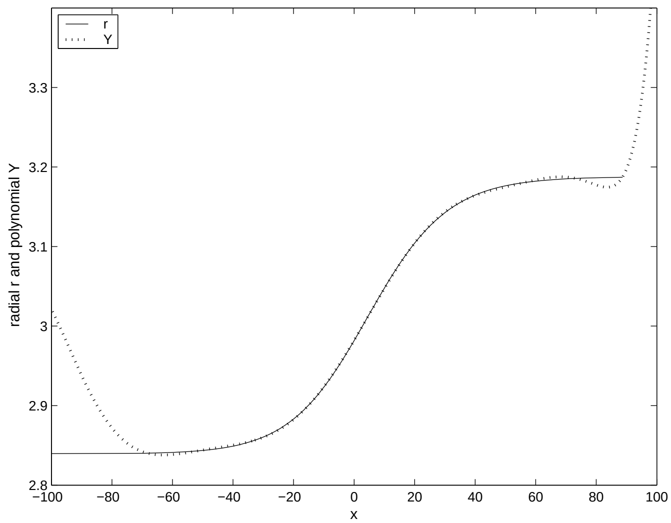

FIG. 2: Radial coordinate r (solid) and its polynomial approximate y (dotted) versus the tortoise coordinate x with $\Lambda = 0.11$.

B. Numerical Solution of the Master Propagating Equation

The radial wave equation (19) and the boundary condition (23) can build up a full boundary value problem. Considering the polynomial approximation (22), we can solve the problem numerically by Mathmatica software. The solution for the first quantum state ($n = 1$) is illustrated in Fig. 3 and Fig. 4. And the other states can be obtained by the same way.

Fig. 3 illustrates the behavior of the field amplitude between the horizons when x is the independent variable. It is shown that the solution is close to a harmonic wave. With r as independent variable, Fig. 4 shows that the waves pile up and the wavelength goes to zero near the two horizons.

Because of the compact property of the tortoise coordinate, it is natural that the waves pile up near the both horizons. The different behavior of the waves near the two horizons in [22] is caused mainly by the imprecise tangent approximate method. Since the same approximate method is used in [27], we can compare our Fig. 2 directly with the Fig. 3 of [27] and conclude that our fitting effect is more precise. Our result shows that the densities of the wave near the two horizons of Fig. 4 are nearly equal in both horizons.

VI. CONCLUSION

The 5D Ricci-flat SdS black string space is derived from a 5D Ricci flat manifold in the STM scenario. The construction of this space is that two 3-branes are embedded into a empty bulk like RS 2-brane model [2]. But unlike the slab of Anti-de Sitter (AdS) space in the standard Randall-Sundrum two branes system, there is no cosmological constant but only the induced (effective) one in this 5D bulk. It is worthy mentioning that the quantum potential labeled by the quantum number n is one of the important results in this black string space. The height and the thickness of the potential are bigger with the increasing quantum number. We assume that the massless scalar field without charge can propagate free in the bulk, and the two branes are stabilized by this scalar field. Furthermore, we consider the Nariai black hole, which is one of the important cases in the singular spaces. In this paper, we employ a more accurate fitting function— polynomial approximation to fit the radial coordinate. With this kind of

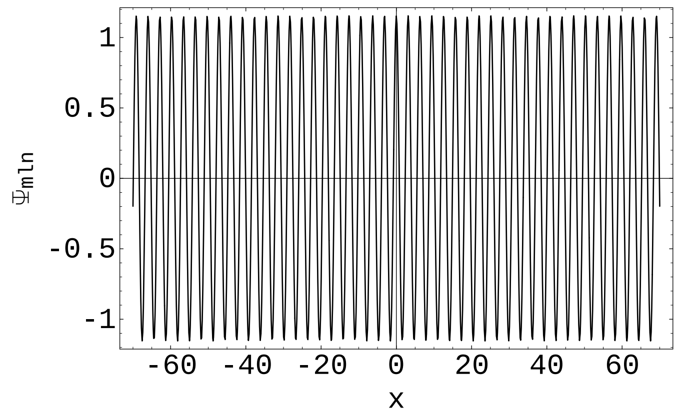

FIG. 3: Variation of the field amplitude versus x with $M = 1, \omega = 1, l = 1, \Lambda = 0.11, y_1 = 10^{\frac{3}{2}}$ and $n = 1$. The solution is close to a harmonic wave.

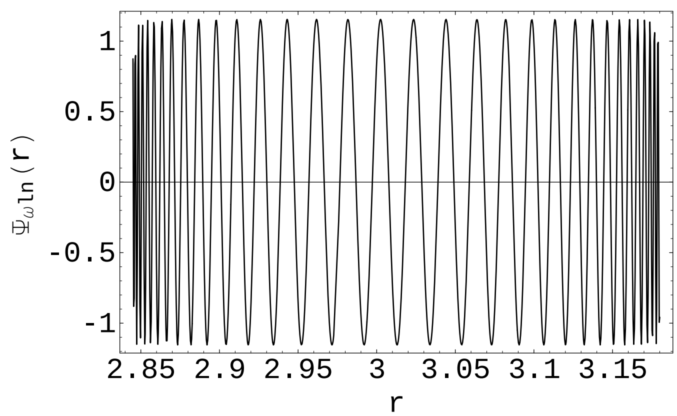

FIG. 4: Variation of the field amplitude versus r with $M = 1, \omega = 1, l = 1, \Lambda = 0.11, y_1 = 10^{\frac{3}{2}}$ and $n = 1$. The waves pile up equally near both horizons.

approximation, a full boundary value problem is formed and solved successfully. Because of the complexity of the differential equation (19), only the numerical solution is presented here. Comparing with the previous results in [22], we can find that the polynomial approximation is more reasonable and the waves pile up almost equally near both horizons.

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